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A NEW SUBCLASS OF MULTIVALENT HARMONIC FUNCTIONS DEFINED BY USING NOVEL INTEGRAL OPERATOR

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Abstract: In this paper a subclass of p-valent (or multivalent) harmonic functions defined by using Novel Integral Operator in the open unit disc has been introduced and some properties as coefficients estimate,extreme points, distortion bounds,closure theorems convex and closure properties have been studied.

Keywords and Phrases: Open Unit disk, Multivalent Functions, Harmonic Functions, Convolution, Novel Integral Operator, Convex set.

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1. Introduction

$$f(z) = z^p + \sum_{k=1}^{\infty} c_{k+p} z^{k+p} (p \in N) \tag{1.1}$$

which are analytic in the open unit disk D=$\{z \in C : |z| < 1\}$.
A continuous function F = H + iG is a complex valued harmonic function in a complex domain C , if both U and V are real harmonic in C. In any simply connected domain $D \subseteq C$,we can write F=H+$\bar{G}$,. We call H the analytic part and G the co-analytic part of F. A necessary and sufficient condition for F to be locally univalent and sense-preserving in D is that $|H'(z)| > |G'(z)|$ in D (see Clunie and Sheil-Small [6]).

Denote by C(p) the class of functions F=H+$\bar{G}$, that are harmonic multivalent and sense-preserving in the unit disk D=$\{z \in C : |z| < 1\}$. The class C(p) was studied by Ahuja and Jahangiri [1] and class C(p) for p=1 was defined and studied by Jahangiri et. al. in [8]. For F=H+$\bar{G} \in C(p)$, we may express the analytic functions H and G as:

$$H(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, G(z) = \sum_{k=0}^{\infty} b_{k+p} z^{k+p} (p \in N) \tag{1.2}$$

where $z \in D$,$|b_p| < 1$.
Now for f(z) of the form (1.1), Lasode etal. [9] have defined the Novel integral operator $I_{b,u}^{n,t}$ for p-valent (or multivalent) functions is as

$$I_{b,u}^{0,t} f(z) = f(z)$$

$$I_{b,u}^{1,t} f(z) = \frac{1}{tz^{(\frac{1}{t}+b-u-1)}} \int_0^z \zeta^{(\frac{1}{t}+b-u-1)} [(tb - tu)\zeta + f(\zeta)] d\zeta = L_t f(z), \quad (t > 0)$$

$$I_{b,u}^{2,t} f(z) = L_t(I_{b,u}^{1,t} f(z))$$

$$I_{b,u}^{3,t} f(z) = L_t(I_{b,u}^{2,t} f(z)).$$

which, in general, means
$I_{b,u}^{n,t} f(z) = L_t(I_{b,u}^{n-1,t} f(z))$.
or in general, it is easily seen that
$I_{b,u}^{n,t} f(z) = z^p + \sum_{k=1}^{\infty} \frac{1}{[1 + (k + p + b - u - 1)t]^n} c_{k+p} z^{k+p}$
Let $I_{b,u}^{n,t} f(z) = z^p + \sum_{k=1}^{\infty} \psi^n(k, p, b, t, u) c_{k+p} z^{k+p}$
where

$$\psi^n(k, p, b, t, u) = \frac{1}{[1 + (k + p + b - u - 1)t]^n} \tag{1.3}$$

$(b, t \geq 0, u \in [0, b], p \in N, n \in N_0, z \in D)$.

Remark. *1.* $I_{b,u}^{0,t} f(z) = I_{b,u}^{n,0} f(z) = I_{b,u}^{0,0} f(z) = f(z) \in C(1)$ [2] *where C(1) is the class of functions of the form*

$$f(z) = z + \sum_{k=1}^{\infty} c_{k+1} z^{k+1} (k \in N) \tag{1.4}$$

2. $I_{b,b}^{n,1} f(z) = I_{u,u}^{n,1} f(z) = I^n f(z)$ *is the Salagean integral operator* [11, 7] *for p=1.*
3. $I_{b,b}^{n,t} f(z) = I_{u,u}^{n,t} f(z) = I^{n,t} f(z)$ *is the Al-Oboudi-Al-Qahtani integral operator* [3, 4, 10].

Now we define a new subclass $\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$ of Harmonic functions $F(z) \in C(p)$, which satisfy the conditions

$$Re\left\{\frac{(1+\alpha e^{i\phi})z(I_{b,u}^{n,t}F(z))'}{I_{b,u}^{n,t}F(z)} - \alpha e^{i\phi}\right\} \geq \beta, \tag{1.5}$$

for $0 \leq \beta < 1, p \in N, n \in N_0 = N \cup (0)$
or equivalently

$$Re\left\{\frac{(1+\alpha e^{i\phi})[[z(I_{b,u}^{n,t}H(z))'] - \overline{[z(I_{b,u}^{n,t}G(z))']}]}{I_{b,u}^{n,t}H(z) + I_{b,u}^{n,t}G(z)} - \alpha e^{i\phi}\right\} \geq \beta \tag{1.6}$$

For H(z) and G(z) given by (1.2), we obtain

$$I_{b,u}^{n,t}H(z) = z^p + \sum_{k=1}^{\infty} \psi^n(k,p,b,t,u)a_{k+p}z^{k+p} \tag{1.7}$$

$$I_{b,u}^{n,t}G(z) = \sum_{k=0}^{\infty} \psi^n(k,p,b,t,u)b_{k+p}z^{k+p} \tag{1.8}$$

We also let $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta) = \Sigma_{b,u}^{n,t}(k,p,\alpha,\beta) \bigcap C(p)$.
In this paper we determine the Coefficients inequality, Distortion theorem, Extreme points, Closure property, Convolution and Convex properties for the class $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

2. Coefficients Inequality

We will first derive a sufficient condition for the functions of the class $\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

Theorem 2.1. *Let F=H+$\bar{G}$ (H and G being given by 1.2). If*

$$\sum_{k=1}^{\infty}[\{(k+p-\beta)+\alpha(k+p-1)\}|a_{k+p}|+ \\ \{(k+p+\beta)+\alpha(k+p+1)\}|b_{k+p}|]\psi^n(k,p,b,t,u) \tag{2.1}$$

$\leq (p-\beta) - \left\{\frac{(p+\beta)+\alpha(p+1)}{[1+(p+b-u-1)t]^n}\right\}|b_p|,\ 0 \leq \beta < 1$ *then* $F \in \Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

Proof. Let F=H+$\bar{G}$ in (2.1), we get

$$Re\left\{\frac{(1+\alpha e^{i\phi})[[z(I_{b,u}^{n,t}H(z))'] - \overline{[z(I_{b,u}^{n,t}G(z))']}]}{I_{b,u}^{n,t}H(z) + I_{b,u}^{n,t}G(z)} - \alpha e^{i\phi}\right\} = Re\left\{\frac{A(z)}{B(z)}\right\} \tag{2.2}$$

where $A(z) = (1+\alpha e^{i\phi})[[z(I_{b,u}^{n,t}H(z))'] - \overline{[z(I_{b,u}^{n,t}G(z))']}] - \alpha e^{i\phi}[I_{b,u}^{n,t}H(z) + I_{b,u}^{n,t}G(z)]$ and $B(z) = [I_{b,u}^{n,t}H(z) + \overline{I_{b,u}^{n,t}G(z)}]$.

Using the fact that Re $\omega(z) \geq \beta$ if and only if $|1-\beta+\omega| \geq |1+\beta-\omega|$ [5], it is sufficient to show that

$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0.$

Substituting for A(z) and B(z), we get

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)| = |[(1+\alpha e^{i\phi})[[z(I_{b,u}^{n,t}H(z))'] - \overline{[z(I_{b,u}^{n,t}G(z))']}]$$
$$-\alpha e^{i\phi}[I_{b,u}^{n,t}H(z) + \overline{I_{b,u}^{n,t}G(z)}]+(1-\beta)[I_{b,u}^{n,t}H(z) + \overline{I_{b,u}^{n,t}G(z)}]|-|[(1+\alpha e^{i\phi})[[z(I_{b,u}^{n,t}H(z))']$$
$$--\overline{[z(I_{b,u}^{n,t}G(z))']}] - \alpha e^{i\phi}[I_{b,u}^{n,t}H(z) + \overline{I_{b,u}^{n,t}G(z)}] - (1+\beta)[I_{b,u}^{n,t}H(z) + \overline{I_{b,u}^{n,t}G(z)}]|$$
$$= |(1+\alpha e^{i\phi})\left[pz^p + \textstyle\sum_{k=1}^{\infty}(k+p)\psi^n(k,p,b,t,u)a_{k+p}z^{k+p}\right.$$
$$\left. - \textstyle\sum_{k=0}^{\infty}(k+p)\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right]$$
$$- \alpha e^{i\phi}\left[z^p + \textstyle\sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^{k+p} + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right] +$$
$$(1-\beta)\left[z^p + \textstyle\sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^{k+p} + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right] | -$$
$$|(1+\alpha e^{i\phi})\left[pz^p + \textstyle\sum_{k=1}^{\infty}(k+p)\psi^n(k,p,b,t,u)a_{k+p}z^{k+p}\right.$$
$$\left. - \textstyle\sum_{k=0}^{\infty}(k+p)\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right]$$
$$- \alpha e^{i\phi}\left[z^p + \textstyle\sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^{k+p} + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right] -$$
$$(1+\beta)\left[z^p + \textstyle\sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^{k+p} + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right] |$$
$$= | \left[(p+1-\beta) + \alpha(p-1)e^{i\phi}\right] z^p +$$
$$\textstyle\sum_{k=1}^{\infty}[(k+p+1-\beta) + \alpha(k+p-1)e^{i\phi}]\psi^n(k,p,b,t,u)a_{k+p}z^{k+p}$$
$$- \textstyle\sum_{k=0}^{\infty}[(k+p-1+\beta) + \alpha(k+p+1)e^{i\phi}]\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}|$$
$$-| \left[(p-1-\beta) + \alpha(p-1)e^{i\phi}\right] z^p +$$
$$\textstyle\sum_{k=1}^{\infty}[(k+p-1-\beta) + \alpha(k+p-1)e^{i\phi}]\psi^n(k,p,b,t,u)a_{k+p}z^{k+p}$$
$$- \textstyle\sum_{k=0}^{\infty}[(k+p+1+\beta) + \alpha(k+p+1)e^{i\phi}]\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}|$$
$$\geq 2(p-\beta)|z|^p - 2\textstyle\sum_{k=1}^{\infty}\left[(k+p-\beta) + \alpha(k+p-1)\right]\psi^n(k,p,b,t,u)|a_{k+p}||z|^{k+p}$$
$$-2\textstyle\sum_{k-0}^{\infty}\left[(k+p+\beta) + \alpha(k+p+1)\right]\psi^n(k,p,b,t,u)|b_{k+p}||z|^{k+p}$$
$$\geq 2\left[(p-\beta) - \left\{\frac{(p+\beta)+\alpha(p+1)}{[1+(p+b-u-1)t]^n}\right\}|b_p|\right]|z|^p$$
$$-2\textstyle\sum_{k=1}^{\infty}\left[(k+p-\beta) + \alpha(k+p-1)\right]\psi^n(k,p,b,t,u)|a_{k+p}||z|^{k+p}$$
$$-2\textstyle\sum_{k=1}^{\infty}\left[(k+p+\beta) + \alpha(k+p+1)\right]\psi^n(k,p,b,t,u)|b_{k+p}||z|^{k+p} \geq 0.$$

Since $|z| < 1$ so let $z \longrightarrow 1^-$. By virtue of (2.1), this implies that $F \in \Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

Theorem 2.2. *Let $F = H + \bar{G} \in C(p)$. Then $F \in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}[\{(k+p-\beta)+\alpha(k+p-1)\}|a_{k+p}| + \{(k+p+\beta)+\alpha(k+p+1)\}|b_{k+p}|]\psi^n(k,p,b,t,u) \leq (p-\beta) - \left\{\frac{(p+\beta)+\alpha(p+1)}{[1+(p+b-u-1)t]^n}\right\}|b_p|, \quad 0 \leq \beta < 1. \tag{2.3}$$

Proof. Since $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta) \subset \Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$, we only need to prove 'only if' part of the theorem. To this end, for functions f of the form (1.2) with condition (1.4), we notice the condition

$$Re\left\{\frac{(1+\alpha e^{i\phi})[[z(I_{b,u}^{n,t}H(z))'] - \overline{[z(I_{b,u}^{n,t}G(z))']}]}{I_{b,u}^{n,t}H(z) + I_{b,u}^{n,t}G(z)} - (\alpha e^{i\phi}+\beta)\right\} \geq 0. \tag{2.4}$$

The above inequality is equivalent to
$Re[|(1+\alpha e^{i\phi})\left[pz^p + \sum_{k=1}^{\infty}(k+p)\psi^n(k,p,b,t,u)a_{k+p}z^{k+p}\right.$
$\left. - \sum_{k=0}^{\infty}(k+p)\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right]$
$-(\alpha e^{i\phi}+\beta)\left[z^p + \sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^{k+p} + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right] \times$
$\left\{z^p + \sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^{k+p} + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^{k+p}}\right\}^{-1}] \geq 0.$
or
$Re[\{p+\alpha(p-1)e^{i\phi}-\beta)\} + \sum_{k=1}^{\infty}\{(k+p-\beta)+\alpha(k+p-1)e^{i\phi}\}\psi^n(k,p,b,t,u)$
$a_{k+p}z^k - \frac{\overline{z^p}}{z^p}\sum_{k=0}^{\infty}(k+p)\{(k+p+\beta)+\alpha(k+p+1)e^{i\phi}\}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^k}]$
$\times\left\{1 + \sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}z^k + \frac{\overline{z^p}}{z^p}\sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)\overline{b_{k+p}z^k}\right\}^{-1} \geq 0.$
Thus condition must hold for all values of z, such that $|z| = r < 1$. Choosing ϕ according to the condition and noting that $Re\{-\alpha e^{\phi}\} \geq -\alpha|e^{i\psi}| = -\alpha$,the above inequality reduces to

$$|\{p+\alpha(p-1)-\beta\} - \left\{\frac{(p+\beta)+\alpha(p+1)}{[1+(p+b-u-1)t]^n}\right\}|b_p|] - \sum_{k=1}^{\infty}[\{(k+p-\beta)+\alpha(k+p-1)\}$$

$$|a_{k+p}| + \{(k+p+\beta)+\alpha(k+p+1)\}|b_{k+p}|]\psi^n(k,p,b,t,u)r^k$$

$$|\left\{1 - \sum_{k=1}^{\infty}\psi^n(k,p,b,t,u)a_{k+p}r^k + \sum_{k=0}^{\infty}\psi^n(k,p,b,t,u)|b_{k+p}|r^k\right\}^{-1}| \geq 0 \tag{2.5}$$

Letting $r \longrightarrow 1^-$ and if the condition (2.3) does not hold, then the numerator in (2.5) is negative. This contradicts our assumptions that $F \in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$. Hence $F \in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.
For $\sum_{k=1}^{\infty}|X_{k+p}| + \sum_{k=0}^{\infty}|Y_{k+p}| = 1$, the harmonic univalent function

$$F(z) = z^p + \sum_{k=1}^{\infty} \frac{(p-\beta)}{\{(k+p-\beta)+\alpha(k+p-1)\}\,\psi^n(k,p,b,t,u)} X_{k+p} z^{k+p}$$
$$+ \sum_{k=0}^{\infty} \frac{(p-\beta)}{\{(k+p+\beta)+\alpha(k+p+1)\}\,\psi^n(k,p,b,t,u)} \overline{Y_{k+p} z^{k+p}} \tag{2.6}$$

shows that equality in the coefficient bound given by (2.3) is sharp.

Corollary 2.3. *Let F=H+$\bar{G} \in C(1)$. Then $F \in C\Sigma_{b,u}^{n,t}(k,1,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty} \left[\{(k+1-\beta)+\alpha k\}\,|a_{k+1}| + \{(k+1+\beta)+\alpha(k+2)\}\,|b_{k+1}|\right] \psi^n(k,1,b,t,u)$$
$$\leq (1-\beta) - \left\{\frac{(1+\beta)+2\alpha}{[1+(b-u)t]^n}\right\} |b_1|, \quad 0 \leq \beta < 1. \tag{2.7}$$

where

$$\psi^n(k,1,b,t,u) = \frac{1}{[1+(k+b-u)t]^n} \tag{2.8}$$

$(b,t \geq 0, u \in [0,b], p \in N, n \in N_0, z \in D)$.

Corollary 2.4. *Let F=H+$\bar{G} \in C(1)$. Then $F \in C\Sigma_{u,u}^{n,t}(k,1,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty} \left[\{(k+1-\beta)+\alpha k\}\,|a_{k+1}| + \{(k+1+\beta)+\alpha(k+2)\}\,|b_{k+1}|\right] \psi^n(k,1,u,t,u)$$
$$\leq (1-\beta) - \{(1+\beta)+2\alpha\}\,|b_1|, \quad 0 \leq \beta < 1. \tag{2.9}$$

where

$$\psi^n(k,1,u,t,u) = \frac{1}{[1+kt]^n} \tag{2.10}$$

$(b,t \geq 0, u \in [0,b], p \in N, n \in N_0, z \in D)$

3. Distribution Theorem

Theorem 3.1. *Let the function $F(z)$=$H(z)$+$\bar{G}(z)$ defined by (1.2) be in the class $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$|F(z)| \leq (1+|b_p|)r^p + [1+(p+b-u)t]^n\left[(p-\beta) - \frac{[(p+\beta)+\alpha(p+1)]}{[1+(p+b-u-1)t]^n}|b_p|\right] r^{p+1} \tag{3.1}$$

and

$$|F(z)| \geq (1-|b_p|)r^p - [1+(p+b-u)t]^n[(p-\beta) - \frac{[(p+\beta)+\alpha(p+1)]}{[1+(p+b-u-1)t]^n}|b_p|]r^{p+1}. \tag{3.2}$$

The result is sharp for the function (2.6).

Proof. We prove only the left hand inequality. Let $F \in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$. Taking the absolute value of F(z), we have

$|F(z)| \geq (1-|b_p|)r^p - \sum_{k=1}^{\infty}\{|a_{k+p}|+|b_{k+p}|\}\, r^{k+p}$

$\geq (1-|b_p|)r^p - r^{p+1}\sum_{k=1}^{\infty}\{|a_{k+p}|+|b_{k+p}|\}$

$\geq (1-|b_p|)r^p - \{[1+(p+b-u-1)t]^n\}^n\, r^{p+1}\sum_{n=p+1}^{\infty}[\{(k+p-\beta)+\alpha(k+p-1)\}$
$|a_{k+p}| + \{(k+p+\beta)+\alpha(k+p+1)\}\,|b_{k+p}|]\psi^n(k,p,b,t,u)$

$\geq (1-|b_p|)r^p - [1+(p+b-u-1)t]^n[(p-\beta) - \dfrac{[(p+\beta)+\alpha(p+1)]}{[1+(p+b-u-1)t]^n}|b_p|]r^{p+1}.$

4. Extreme Points

Theorem 4.1. *A function $F = H + \bar{G} \in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$ if and only if F(z) can be expressed in the form*

$$F(z) = \sum_{k=0}^{\infty}(A_{k+p}H_{k+p} + B_{k+p}G_{k+p}) \tag{4.1}$$

where

$$H_p(z) = z^p, H_{k+p}(z) = z^p + \frac{\{p-\beta\}}{\{(k+p-\beta)+\alpha(k+p-1\}\,\psi^n(k,p,b,t,u)}z^{k+p}(k\geq 1) \tag{4.2}$$

$$G_{k+p}(z) = z^p + \frac{\{p-\beta\}}{\{(k+p+\beta)+\alpha(k+p+1\}\,\psi^n(k,p,b,t,u)}z^{k+p}(k\geq 0),$$

$\psi^n(k,p,b,t,u)$ is given by (1.3) and $\sum_{k=0}^{\infty}(A_{k+p}+B_{k+p}) = 1$ with $A_{k+p}\geq 0, B_{k+p}\geq 0$. In particular the extreme points of $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$ are H_{k+p} and G_{k+p}.

Proof. Let F(z) be of the form (4.1). Then we have

$F(z) = \sum_{k=0}^{\infty}(A_{k+p}+B_{k+p})\,z^p$

$+\sum_{k=1}^{\infty}\dfrac{\{p-\beta\}}{\{(k+p-\beta)+\alpha(k+p-1\}\,\psi^n(k,p,b,t,u)}A_{k+p}z^{k+p}$

$+\sum_{k=0}^{\infty}\dfrac{\{p-\beta\}}{\{(k+p+\beta)+\alpha(k+p+1\}\,\psi^n(k,p,b,t,u)}B_{k+p}z^{k+p}$

$= z^p + \sum_{k=1}^{\infty}\dfrac{\{p-\beta\}}{\{(k+p-\beta)+\alpha(k+p-1\}\,\psi^n(k,p,b,t,u)}A_{k+p}z^{k+p}$

$+\sum_{k=0}^{\infty}\dfrac{\{p-\beta\}}{\{(k+p+\beta)+\alpha(k+p+1\}\,\psi^n(k,p,b,t,u)}B_{k+p}z^{k+p}$

Furthermore, let

$|a_{k+p}| = \dfrac{\{p-\beta\}}{\{(k+p-\beta)+\alpha(k+p-1\}\,\psi^n(k,p,b,t,u)}A_{k+p}$

and $|b_{k+p}| = \dfrac{\{p-\beta\}}{\{(k+p+\beta)+\alpha(k+p+1\}\,\psi^n(k,p,b,t,u)}B_{k+p}$.

Now, $\sum_{k=1}^{\infty}\left\{\dfrac{\{(k+p+\beta)+\alpha(k+p+1)\}}{(p-\beta)}\right\}\psi^n(k,p,b,t,u)|a_{k+p}|+$

$\sum_{k=0}^{\infty}\left\{\dfrac{\{(k+p+\beta)+\alpha(k+p+1\}}{(p-\beta)}\right\}\psi^n(k,p,b,t,u)|b_{k+p}|$

$=\sum_{k=1}^{\infty}A_{k+p}+\sum_{k=0}^{\infty}B_{k+p}=1-A_p\leq 1$. So $F\in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

Conversely, suppose that $F\in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

Setting $A_{k+p}=\left\{\dfrac{\{(k+p+\beta)+\alpha(k+p+1)\}}{(p-\beta)}\right\}\psi^n(k,p,b,t,u)|a_{k+p}|\ (k\geq 1)$

$B_{k+p}=\left\{\dfrac{\{(k+p+\beta)+\alpha(k+p+1\}}{(p-\beta)}\right\}\psi^n(k,p,b,t,u)|b_{k+p}|\ (k\geq 0)$

and $A_p=1-\sum_{k=1}^{\infty}A_{k+p}-\sum_{k=0}^{\infty}B_{k+p}$.

Therefore,

$F(z)=z^p+\sum_{k=1}^{\infty}|a_{k+p}|z^{k+p}+\sum_{k=0}^{\infty}|b_{k+p}|z^{k+p}$

$=z^p+\sum_{k=1}^{\infty}\dfrac{\{p-\beta\}}{\{(k+p-\beta)+\alpha(k+p-1)\}\,\psi^n(k,p,b,t,u)}A_{k+p}z^{k+p}$

$+\sum_{k=0}^{\infty}\dfrac{\{p-\beta\}}{\{(k+p+\beta)+\alpha(k+p+1)\}\,\psi^n(k,p,b,t,u)}B_{k+p}z^{k+p}$

$=z^p+\sum_{k=1}^{\infty}\left[A_{k+p}\left\{H_{k+p}(z)-z^p\right\}\right]+\sum_{k=0}^{\infty}\left[B_{k+p}\left\{G_{k+p}(z)-z^p\right\}\right]$

So $F(z)=\sum_{k=0}^{\infty}\left(A_{k+p}H_{k+p}(z)+B_{k+p}G_{k+p}(z)\right)$.

5. Convex Property

Theorem 5.1. *The class $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$ is a convex set.*

Proof. Let the function $F_{k+p,j}(j=1,2)$ defined by

$F_{k+p,j}(z)=z^p+\sum_{k=1}^{\infty}|a_{k+p,j}|z^{k+p}+\sum_{k=0}^{\infty}\overline{|b_{k+p,j}|z^{k+p}}$ be in the class $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

It is sufficient to prove that the function

$\mathcal{H}(z)=tF_{k+p,1}(z)+(1-t)F_{k+p,2}(z)\ (0\leq t<1)$ is also in the class $C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

Since for $(0\leq t<1)$,

$\mathcal{H}(z)=z^p+\sum_{k=1}^{\infty}[t|a_{k+p,1}|+(1-t)|a_{k+p,2}|]z^{k+p}+\sum_{k=0}^{\infty}[t|b_{k+p,1}|+(1-t)|b_{k+p,2}|]\overline{z}^{k+p}$.

With the aid of theorem (2.2), we have

$\sum_{k=1}^{\infty}\left[(k+p-\beta)+\alpha(k+p-1)\right]\psi^n(k,p,b,t,u)[t|a_{k+p,1}|+(1-t)|a_{k+p,2}|]$

$+\sum_{k-0}^{\infty}\left[(k+p+\beta)+\alpha(k+p+1)\right]\psi^n(k,p,b,t,u)[t|b_{k+p,1}|+(1-t)|b_{k+p,2}|]$

$=t\sum_{k=1}^{\infty}\left\{(k+p-\beta)+\alpha(k+p-1)\right\}|a_{k+p,1}|+$

$\left\{(k+p+\beta)+\alpha(k+p+1)\right\}|b_{k+p,1}]\psi^n(k,p,b,t,u)+$

$(1-t)\sum_{k=1}^{\infty}[\{(k+p-\beta)+\alpha(k+p-1)\}\,|a_{k+p,2}|+$
$\{(k+p+\beta)+\alpha(k+p+1)\}\,|b_{k+p,2}]\psi^{n}(k,p,b,t,u)$
$\leq t\left\{[(p-\beta)-\frac{[(p+\beta)+\alpha(p+1)]}{[1+(p+b-u-1)t]^{n}}|b_p|]\right\}$
$+(1-t)\left\{[(p-\beta)-\frac{[(p+\beta)+\alpha(p+1)]}{[1+(p+b-u-1)t]^{n}}|b_p|]\right\}$
$\leq \left\{[(p-\beta)-\frac{[(p+\beta)+\alpha(p+1)]}{[1+(p+b-u-1)t]^{n}}|b_p|]\right\}.$
Hence $\mathcal{H}(z)\in C\Sigma_{b,u}^{n,t}(k,p,\alpha,\beta)$.

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